COMPLEX NUMBERS

We often meet equations such as $x^2 + 1 = 0$ whose roots are not real. We cannot take the square root of a negative number.
We need another category of numbers, namely ***the set of imaginary numbers.***

We assume the existence of a "number" i such that $i^2 = -1$.
Then the equation $x^2 + 1 = 0$ can be solved as follows:
$x^2 = -1 \quad \rightarrow \quad x^2 = i^2 \quad \rightarrow \quad x = +i$

and any quadratic equation can now be solved .

Consider the quadratic equation $x^2 - 2x + 10 = 0$.
Let us try to solve it by completing the square:

$x^2 - 2x = -10,$

$x^2 - 2x + 1 = -10 + 1,$

$\left(x - 1\right)^2 = -9 = 9i^2,$

$x - 1 = +3i \qquad \text{or} \qquad x - 1 = -3i,$

$x = 1 + 3i = 0 \qquad \text{or} \qquad x = 1 - 3i.$

We could get this same result by using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{for} \qquad ax^2 + bx + c = 0,$$

i.e. $x = \dfrac{2 \pm \sqrt{-36}}{2} = \dfrac{2 \pm \sqrt{36i^2}}{2} = \dfrac{2 \pm 6i}{2} = 1 \pm 3i.$

This quadratic equation has been regarded as unsolvable up to now because $b^2 - 4ac < 0$.

However, assuming the existence of this "number" i, we can see that any quadratic equation can now be solved . We obtain solutions of the form $a + ib$ and $a - ib$, where a and b are ordinary real numbers.

A general complex number can be written in the form $z = a + ib$, where a and b are real numbers, including zero.
a ***is called the "Real part of z"*** $\Rightarrow$ $\operatorname{Re} z = a$
b ***is called the "Imaginary part of z"*** $\Rightarrow$ $\operatorname{Im} z = b$

The number $a - ib$ is called the conjugate of $a + ib$,

i.e. if $z = a + ib$ then $\overline{z} = a - ib$.

If $a = 0 \Rightarrow z = ib \longrightarrow$ imaginary number.

If $b = 0 \Rightarrow z = a \longrightarrow$ real number.

The field of complex numbers is

$$\mathbb{C} = \left\{ z = a + ib: \quad a, b \in \mathbb{R}, \quad i^2 = -1 \right\}.$$

• Algebraic operations

Let a, b, c, d - real numbers.

1. Addition and Subtraction

$$(a + ib) + (c + id) = (a + c) + i(b + d).$$

2. Multiplication

$$(a + ib)(c + id) = ac + iad + ibc + i^2 bd$$

$$= ac - bd + i(ad + bc).$$

Special case

$(a + ib)(a - ib) = a^2 + b^2$ - **real number!**

3. Division

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac - iad + ibc - i^2 bd}{c^2 + d^2}$$

$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

$$= \left(\frac{ac + bd}{c^2 + d^2} \right) + i \left(\frac{bc - ad}{c^2 + d^2} \right),$$

e.g. multiplying numerator and denominator by the conjugate of the denominator.

Note

Power of i can be simplified:

$i^3 = i^2 i = -i,$

$i^4 = \left(i^2\right)^2 = (-1)^2 = 1,$

$i^5 = i^4 i = i,$

$i^6 = \left(i^2\right)^3 = (-1)^3 = -1,$

and so on.

Worked examples

1) $(3-i)(2+5i) = 6 + 15i - 2i - 5i^2 = 11 + 13i.$

2) $$\frac{-1+2i}{2+3i} = \frac{(-1+2i)}{(2+3i)} . \frac{(2-3i)}{(2-3i)} = \frac{-2+3i+4i-6i^2}{2^2+3^2}$$

$$= \frac{4+7i}{13} = \frac{4}{13} + \frac{7}{13}i.$$

3) $$\frac{1}{(1-i)(3+2i)} = \frac{1}{3+2i-3i-2i^2} = \frac{1}{5-i}$$

$$= \left(\frac{1}{5-i}\right)\left(\frac{5+i}{5+i}\right) = \frac{5+i}{25+1} = \frac{5+i}{26} = \frac{5}{26} + \frac{1}{26}i.$$

• Equal Complex Numbers

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal $\Leftrightarrow$ their real parts are equal and their imaginary parts are equal :

$a + ib = c + id \quad \Leftrightarrow \quad a = c,\ b = d.$

Example. If x and y are real, find x and y such that

$x(4-i) + y(2+3i) = 8 + 5i.$

Solution:

Open parenthesis and compare the real and imaginary parts:

$$4x - ix + 2y + i3y = 8 + 5i,$$

$$(4x + 2y) + i(3y - x) = 8 + 5i,$$

$$\begin{cases} 4x + 2y = 8 \\ 3y - x = 5 \end{cases} \Leftrightarrow \begin{cases} 2x + y = 4 \\ x = 3y - 5 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 2. \end{cases}$$

The method of equating real and imaginary parts of a complex equation also provides a way of determining the square root of a complex number, e.g. to find $\sqrt{3 - 4i}$, we say:

Let $\sqrt{3 - 4i} = x + iy$, where x and y are real.

Then $3 - 4i = (x + iy)^2 = x^2 - y^2 + 2xyi$.

Equating real and imaginary parts gives:

$$\begin{cases} x^2 - y^2 = 3 \\ 2xy = -4 \end{cases} \Leftrightarrow \begin{cases} x = \pm 2 \\ y = \pm 1, \end{cases}$$

so $\sqrt{3 - 4i} = \pm(2 - i)$.

Note

Remember that the imaginary part of a complex number is in fact real - it is the coefficient of i!

• Complex Roots of a quadratic equations

Consider the quadratic equation $ax^2 + bx + c = 0$. (1)

If $D = b^2 - 4ac < 0$,

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Leftrightarrow x = \dfrac{-b}{2a} \pm \dfrac{i\sqrt{4ac - b^2}}{2a}$. (2)

Let $p = \dfrac{-b}{2a}$ and $q = \dfrac{\sqrt{4ac - b^2}}{2a}$,

so the roots of (1) are $p + iq$ and $p - iq$ $\Rightarrow$ conjugate complex numbers.

Example.

$x^2 + 2x + 5 = 0$,

$D = \sqrt{b^2 - 4ac} = 4 - 20 = -16 < 0$,

and so there are no real solutions.

Using the formula (2), we have that

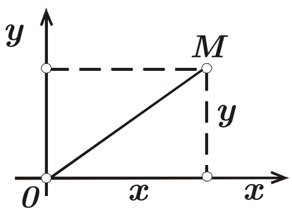
$$x = \frac{-2 \pm \sqrt{-16}}{2.1} = \frac{-2 \pm 4i}{2} = -1 \pm 2i,$$

i.e. $x = -1 + 2i$ or $x = -1 - 2i$.

• The Argand Diagram

The complex number $z = x + iy$, with x and y real numbers, can be represented by the ordered pair (x, y).

This suggests an analogy between complex numbers and the coordinates of a point (x, y) in ordinary Cartesian coordinates.



Clearly, we can think of our complex number z as being represented on a plane as the point M with Cartesian coordinates (x, y) referred to axes Ox and Oy.

Since real numbers are complex numbers of the form $x + i0$, i.e. $y = 0$, it follows that all real numbers lie on the axis Ox.

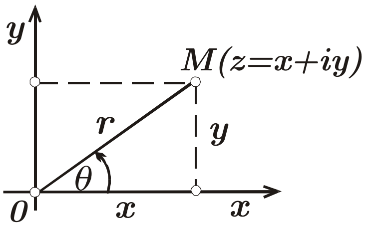
The axis Ox is called the ***Real Axis.***

Similarly - the axis Oy is called the ***Imaginary Axis***.

This idea was introduced by the French mathematician Argand and his name is given to the diagram which represents a complex number in this way – Argand diagram.

A general complex number $z = x + iy$ is represented by the vector $\overrightarrow{OM}$ where M is the point (x, y).

• Modulus and Argument of a complex number



Let the complex number $z = x + iy$ is represented by the point $M(x, y)$.
If $OM = r$ then
r is called the modulus of z
(this is the distance of the p. M from the origin).

$r = |z| = |x + iy| = \sqrt{x^2 + y^2}$ (by Pythagoras).

The angle θ (in the positive sense) is called the argument of z .

We use the notation $\arg z$,
i.e. $\theta = \arg z = \arg(x + iy)$.
From the Argand diagram we see that $\tan\theta = \dfrac{y}{x}$.

Note

1. If x and y are known, there is an infinite set of angles whose tangent is $\dfrac{y}{x}$,
so there is also an infinite set of arguments for $x + iy$.
So, to get the correct value for our problem, always draw a diagram.

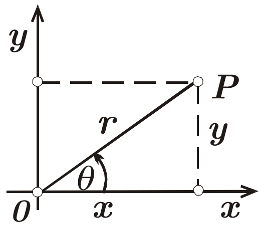
2. There is only one value θ in the range $-\pi < \theta < \pi$.

This value is the principal value of $\arg z$.

The other correct values for $\arg z$ are $\theta \pm 2k\pi,\ k = 0, \pm 1, \pm 2, \ldots$.

• The Polar form of a complex number

Let $z = x + iy$. From the diagram we see that



$$\left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right.$$

Hence

$$x + iy = r\cos\theta + i\sin\theta = r(\cos\theta + i\sin\theta).$$

This is called the Polar form of z

(because r and θ are the polar coordinates of the point P, whose Cartesian coordinates are x and y).

If a complex number is given in the form $x + iy$, it can be converted into the form $r(\cos\theta + i\sin\theta)$ simply by finding the modulus r and the argument θ.

Some problems involving complex numbers are much more easily solved by using the Polar form.

Example.

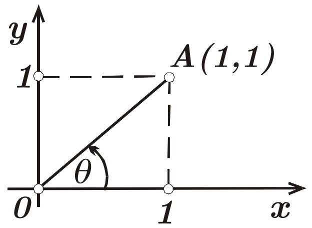
Represent the following complex numbers on Argand diagram and hence express them in Polar form:

a) $1 + i$ **b)** $-3 + 4i$ **c)** -5 **d)** $2i$ **e)** $2 - 2i$

Solution:

a) The point A represents $(1 + i)$.

a)



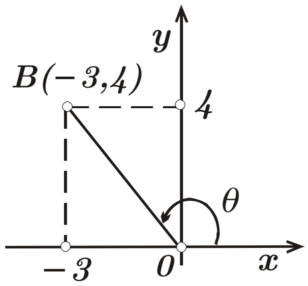
Distance $OA = \sqrt{1^2 + 1^2} = \sqrt{2} = |1 + i| = r$,

$\tan\theta = 1$, and so $\theta = \dfrac{\pi}{4}$.

Thus $1 + i = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$.

b) The point B represents $(-3+4i)$.

b)



Distance

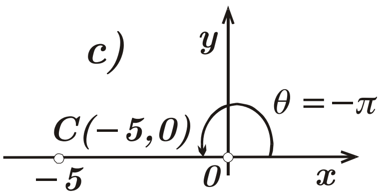
$OB = \sqrt{3^2+4^2} = 5 = |-3+4i| = r.$

$\tan\theta = \dfrac{-4}{3}$, and so $\theta = 180^\circ - 53^\circ 8' = 126^\circ 2'.$

Thus $-3+4i = 5\left(\cos 126^\circ 2' + i\sin 126^\circ 2'\right).$

c) The point C represents -5.

c)



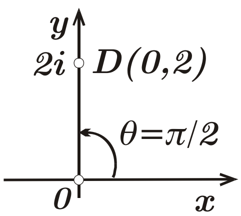
Distance $OC = |-5| = 5 = r$

and $\theta = \pi$.

Thus $-5 = 5(\cos\pi + i\sin\pi)$.

d) The point D represents $2i$.

d)



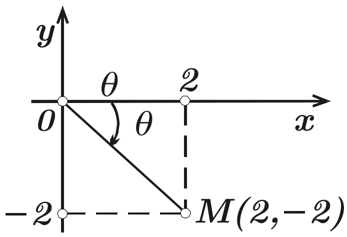
Distance $OD = \sqrt{0^2+2^2} = 2 = |2i| = r$

and $\theta = \dfrac{\pi}{2}$

Thus $2i = 2\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right).$

e) The point M represents $(2-2i)$.

e)



Distance $OM = \sqrt{2^2+2^2} = \sqrt{8} = |2-2i| = r$,

$\tan\theta = -1$, and so $\theta = -\dfrac{\pi}{4}$.

Thus $2-2i = \sqrt{8}\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right).$

•Product and Quotient in Polar form

Let $z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$
and $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$
Then
$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$
i.e. $z_1 z_2$ gives a complex number with modulus $r_1 r_2$ and argument $(\theta_1 + \theta_2)$.

We also find that
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$
i.e. $\frac{z_1}{z_2}$ gives a complex number with modulus $\frac{r_1}{r_2}$ and argument $(\theta_1 - \theta_2)$.

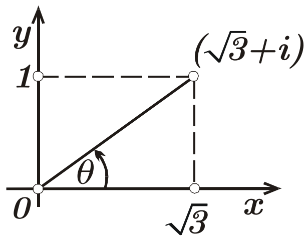
These results may be easily remembered in words:

To multiply: multiply the modulus and add the arguments.

To divide: divide the modulus and subtract the arguments.

Example 1. Simplify $\left(\sqrt{3} + i\right)^{18}$.

We shell use the Polar form as follows



$$|z| = r = \sqrt{\sqrt{3}^2 + 1^2} = 2;$$
$$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow \arg z = \frac{\pi}{6};$$
$$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$

From $|z| = 2 \Rightarrow |z|^{18} = 2^{18}$.

From $\arg z = \frac{\pi}{6} \Rightarrow \arg(z)^{18} = 18\left(\frac{\pi}{6}\right) = 3\pi$.

Thus we obtain

$$\begin{aligned}(\sqrt{3}+i)^{18} &= 2^{18}(\cos 3\pi + i\sin 3\pi) \\ &= 2^{18}(\cos\pi + i\sin\pi) \\ &= 2^{18}(-1+0i) \\ &= -2^{18}.\end{aligned}$$

Hence $z^n = r^n(\cos n\theta + i\sin n\theta)$.

Example 2. Simplify $\dfrac{(1+i)^8}{(\sqrt{3}-i)^6}$.

From the diagrams we see that

$$(1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

$$(\sqrt{3}-i) = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right).$$

Hence

$$(1+i)^8 = \sqrt{2}^8\left(\cos\left(8\,\frac{\pi}{4}\right) + i\sin\left(8\,\frac{\pi}{4}\right)\right) = 2^4(\cos 2\pi + i\sin 2\pi),$$

$$(\sqrt{3}-i)^6 = \sqrt{3}^6\left(\cos 6\left(-\frac{\pi}{6}\right) + i\sin 6\left(-\frac{\pi}{6}\right)\right)$$

$$= 2^6(\cos(-\pi) + i\sin(-\pi)).$$

So we get

$$\begin{aligned}\frac{(1+i)^8}{(\sqrt{3}-i)^6} &= \frac{2^4}{2^6}(\cos(2\pi-(-\pi)) + i\sin(2\pi-(-\pi))) \\ &= \frac{1}{4}(\cos 3\pi + i\sin 3\pi) \\ &= \frac{1}{4}(-1+0i) \\ &= -\frac{1}{4}.\end{aligned}$$

• Exponential form of a complex number

From the Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, and the Polar form of a complex number
$z = r(\cos\theta + i\sin\theta)$,
we get $z = r\,e^{i\theta}$ - ***Exponential form of*** z.

Example.

Express $1+i$ in $z = r\,e^{i\theta}$ form and hence find $(1+i)^{20}$ in *Cartesian form.*

Solution:

We found that the Polar form of this complex number is
$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$,
i.e. $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$.

So the Exponential form will be $1+i = \sqrt{2}\,e^{i\frac{\pi}{4}}$.
Hence

$$(1+i)^{20} = \sqrt{2}^{20}\,e^{i\frac{\pi}{4}20} = 2^{10}\,e^{i5\pi} = 2^{10}(\cos 5\pi + i\sin 5\pi)$$
$$= 2^{10}(-1+0i)$$
$$= -2^{10}.$$

Note

We now have three ways of expressing a complex number:
$z = x + iy$ - ***Cartesian form.***
$z = r(\cos\theta + i\sin\theta)$ - ***Polar form.***
$z = r\,e^{i\theta}$ - ***Exponential form.***

• Exercises I

1) Solve the equations

a) $z^2 + 9 = 0$ *Ans.* $z = \pm 3i$.

b) $z^2 + 4z + 8 = 0$ *Ans.* $z = -2 \pm 2i$.

c) $z^2 - 3z + 3 = 0$ *Ans.* $z = \dfrac{3 \pm i\sqrt{3}}{2}$.

d) $z^4 - 3z^2 - 4 = 0$ *Ans.* $z = \pm 2;\ \pm i$.

2) Given $z = -1 + 3i$, express $z + \dfrac{2}{z}$ in Cartesian form $a + ib$.

3) Given $z = 4 - 3i$, express $z + \dfrac{1}{z}$ in Cartesian form $a + ib$.

4) Express in $a + ib$ form.

a) $\dfrac{(1 - i)}{(3 - i)^2}$ **b)** $\dfrac{(2 - 3i)}{(1 + 2i)\, i^3}$ **c)** $\dfrac{(1 - 2i)(3 + i)}{(4 + i)}$ **d)** $\dfrac{(1 + i)\, i^5}{(2 - i)^2}$.

5) Find z, given that $z(2 - 3i) = 3 + 4i$.

6) Simplify $(3 + 2i)^3 - (3 - 2i)^3$.

7) If $z_1 = \dfrac{2 - i}{2 + i}$, $z_2 = \dfrac{2i - 1}{1 - i}$, express z_1 and z_2 in $a + ib$ form.

8) If $z_1 = -1 + 2i$, $z_2 = 3 - 4i$, $z_3 = 1 - i$, find

$\left(\dfrac{1}{z_1} - \dfrac{1}{z_2}\right)$ and $\left(z_2 + \dfrac{1}{z_3}\right)$ in $a + ib$ form.

9) If $z_1 = 1 - 3i, \; z_2 = -1 - i, \; z_3 = 1 - i\sqrt{3}$

find $\left(\frac{1}{z_1} + \frac{1}{z_2}\right)$ and $\left(z_3 - \frac{1}{z_1}\right)$ in $a + ib$ form.

• Exercises II

1) Find the modulus and arguments of the complex numbers:

a) -4 **b)** $-3i$ **c)** $-1 + i$ **d)** $2 + 2i$

f) $1 + i\sqrt{3}$ **g)** $-\sqrt{3} - i$ **h)** $5i$ **k)** $-1 - i$.

2) Find the Polar form of 1; $1 + i$; $1 - i$; $-3 + 4i$; $2 + i$.
Hence, find the modulus and arguments of :
a) $\frac{1}{-3 + 4i}$ **b)** $(1 + i)(1 - i)$ **c)** $\frac{i}{-3 + 4i}$ **d)** $\frac{2 + i}{1 - i}$.

3) Express $1 - i$ and $1 - i\sqrt{3}$ in Polar form and hence express the following in Cartesian form:
a) $(1 - i)^{16}$ **b)** $(1 - i\sqrt{3})^{18}$ **c)** $\left(\frac{1 - i}{1 - i\sqrt{3}}\right)^{12}$.

4) If $z = 4 - 3i$, express $\left(z + \frac{1}{z}\right)$ in $a + ib$ form.

5) Express $z = -1 + i\sqrt{3}$ in Polar form and hence find $(z)^{12}$.

6) Express $z = \sqrt{3} - i$ in Polar form and hence find $(z)^{18}$.

7) Express $z = \sqrt{3} + i$, $z = -1 - i$, $z = 2 - 2i$, $z = -1 + i\sqrt{3}$ in Exponential form.

8) Solve the equation $z^2 + 2z + 5 = 0$.

9) Given that z and $\bar{z}$ satisfy the equation $z\bar{z} + 2iz = 12 + 6i$, find the possible values of z.

10) Find $\sqrt{4-3i}$;

11) If $z=\frac{2+i}{1-i}$ find the real and imaginary parts of $z+\frac{1}{z}$.

12) If $\frac{(2-i)(3+2i)}{3-4i}=r(\cos\theta+i\sin\theta)$, where r and θ are real,

show that $r=\sqrt{12}$ and $\tan\theta=\frac{7}{4}$.

13) Express $1+\sqrt{3}i$ and $-1-i$ in polar form and hence find $\frac{(1+\sqrt{3}i)^9}{(-1-i)^4}$

in Cartesian form.

14) If $z_1=\cos\theta+i\sin\theta$ and $z_2=\cos\varphi+i\sin\varphi$ prove that

a) $\frac{1+z_1}{1-z_1}=i\cot\frac{\theta}{2}$

b) $\frac{1}{2}\left(z_1 z_2+\frac{1}{z_1 z_2}\right)=\cos(\theta+\varphi)$